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One Diophantine inequality with unlike powers of prime variables

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Abstract

In this paper, we show that if $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are nonzero real numbers not all of the same sign, η is real, $0 < \sigma < \frac{1}{720}$, and at least one of the ratios λ_i/λ_j $(1 \le i < j \le 5)$ is irrational, then the inequality $|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < (\max_{1\le j\le 5} p_j^j)^{-\sigma}$ has infinite solutions with primes p_1, p_2, p_3, p_4, p_5.

MSC: 11D75; 11P55

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1 Introduction

Diophantine inequalities with integer or prime variables have been considered by many scholars. Recently, Yang and Li in [1] proved that the inequality

$$\left|\lambda_1 x_1^2 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^5 - p - \frac{1}{2}\right| < \frac{1}{2}$$

has infinite solutions with natural numbers x_1, x_2, x_3, x_4 and prime p. Using the Davenport-Heilbronn method, we establish our result as follows.

Theorem 1.1 *Let λ_1, λ_2, λ_3, λ_4, λ_5 be nonzero real numbers not all of the same sign, η is real, $0 < \sigma < \frac{1}{720}$, and at least one of the ratios λ_i/λ_j $(1 \le i < j \le 5)$ is irrational, then the inequality*

$$\left|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta\right| < \left(\max_{1\le j\le 5} p_j^j\right)^{-\sigma}$$

has infinite solutions with primes p_1, p_2, p_3, p_4, p_5.

2 Notation and outline of the proof

Throughout, we use p to denote a prime number. We denote by δ a sufficiently small positive number and by ε an arbitrarily small positive number, not necessarily the same at different occurrences. Constants, both explicit and implicit, in Landau or Vinogradov symbols may depend on λ_1, λ_2, λ_3, λ_4, λ_5, and η. We write $e(x) = e^{2\pi ix}$. We take X to be the basic parameter, a large real integer. Since at least one of the ratios λ_i/λ_j $(1 \le i < j \le 5)$ is irrational, without loss of generality we may assume that λ_1/λ_2 is irrational. For the other cases, the only difference is in the following intermediate region, and we may deal with the same method in Section 4.

Since λ_1/λ_2 is irrational, there are infinitely many pairs of integers q, a with $|\lambda_1/\lambda_2 - a/q| \leq q^{-2}$, $(a,q) = 1$, $q > 0$, and $a \neq 0$. We choose q to be large in terms of λ_1, λ_2, λ_3, λ_4, λ_5, η and make the following definitions:

$$N = q^2, \qquad L = \log N, \qquad 0 < \sigma < \frac{\theta}{32} < \frac{1}{720}, \nu = N^{-\sigma}, \qquad \tau = N^{-1+\theta}, \tag{2.1}$$

$$P = N^{\theta}L^{-1}, \qquad Q = \left(|\lambda_1|^{-1} + |\lambda_2|^{-1}\right)N^{1-\theta}, \qquad T_1 = T_2^2 = T_3^3 = T_4^4 = T_5^5 = N^{\frac{1}{3}}. \tag{2.2}$$

Let u be a positive real number, we define

$$K_u(\alpha) = \left(\frac{\sin \pi u\alpha}{\pi\alpha}\right)^2 \quad (\alpha \neq 0), \qquad K_u(0) = u^2, \tag{2.3}$$

$$F_k(\alpha) = \sum_{(\delta N)^{1/k} \leq p \leq N^{1/k}} e\left(\lambda_k p^k \alpha\right) \log p, \quad k = 1,2,3,4,5, \tag{2.4}$$

$$I_k(\alpha) = \int_{(\delta N)^{1/k}}^{N^{1/k}} e\left(\lambda_k y^k \alpha\right) \mathrm{d}y, \quad k = 1,2,3,4,5, \tag{2.5}$$

$$J_k(\alpha) = \sum_{\substack{|\gamma| \leq T_k \\ \beta \geq \frac{2}{3}}} \sum_{\delta N < n \leq N} n^{-1+\rho/k} e(\lambda_k \alpha n), \quad k = 1,2,3,4,5, \tag{2.6}$$

where $\rho = \beta + i\gamma\,(\beta, \gamma$ real) is a typical non-trivial zero of the Riemann Zeta function.

It follows from (2.3) that

$$K_u(\alpha) \ll \min\left(u^2, |\alpha|^{-2}\right), \qquad \int_{-\infty}^{+\infty} e(\alpha y) K_u(\alpha)\,\mathrm{d}\alpha = \max\left(0, u - |y|\right). \tag{2.7}$$

From (2.7) it is clear that

$$\begin{aligned} J &:= \int_{-\infty}^{+\infty} \prod_{j=1}^{5} F_j(\alpha) e(\alpha\eta) K_\nu(\alpha)\,\mathrm{d}\alpha \\ &\leq (\log N)^5 \sum_{\substack{|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < \nu \\ (\delta N)^{1/k} \leq p_k \leq N^{1/k}, k=1,2,3,4,5}} 1 \\ &=: (\log N)^5 \mathcal{N}(N). \end{aligned}$$

Thus we have

$$\mathcal{N}(N) \geq (\log N)^{-5} J.$$

To estimate J, we split the range of infinite integration into three sections, traditional named the neighborhood of the origin $\mathfrak{C} = \{\alpha \in \mathbb{R} : |\alpha| \leq \tau\}$, the intermediate region $\mathfrak{D} = \{\alpha \in \mathbb{R} : \tau \leq |\alpha| \leq P\}$, the trivial region $\mathfrak{c} = \{\alpha \in \mathbb{R} : |\alpha| > P\}$.

To prove Theorem 1.1, we shall establish that

$$J(\mathfrak{C}) \gg \nu^2 N^{\frac{77}{60}}, \qquad J(\mathfrak{D}) = o\left(\nu^2 N^{\frac{77}{60}}\right), \qquad J(\mathfrak{c}) = o\left(\nu^2 N^{\frac{77}{60}}\right)$$

in Sections 3, 4, and 5, respectively. Thus

$$\mathcal{N}(N) \gg \nu^2 (\log N)^{-5} N^{\frac{77}{60}},$$

and Theorem 1.1 can be established.

3 The neighborhood of the origin

We let

$$B_k(\alpha) = F_k(\alpha) - I_k(\alpha) + J_k(\alpha), \quad k = 1,2,3,4,5. \tag{3.1}$$

We use C to denote a positive absolute constant, not necessarily the same one on each occurrence.

Lemma 3.1 *We have*

$$B_k(\alpha) \ll N^{\frac{2}{3k}} L^C \bigl(1 + |\alpha|N\bigr), \quad k = 1,2,3,4,5. \tag{3.2}$$

This is Lemma 7 *of Vaughan* [2].

Lemma 3.2 *For* $k = 1,2,3,4,5$, *we have*

$$I_k(\alpha) \ll N^{\frac{1}{k}} \min\bigl(1, N^{-1}|\alpha|^{-1}\bigr), \tag{3.3}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \bigl|J_k(\alpha)\bigr|^2 \,\mathrm{d}\alpha \ll N^{\frac{2}{k}-1} \exp\bigl(-2L^{-\frac{1}{5}}\bigr), \tag{3.4}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \bigl|I_k(\alpha)\bigr|^2 \,\mathrm{d}\alpha \ll N^{\frac{2}{k}-1}, \tag{3.5}$$

$$\int_{-\tau}^{\tau} \bigl|B_k(\alpha)\bigr|^2 \,\mathrm{d}\alpha \ll N^{\frac{2}{k}-1} \exp\bigl(-2L^{-\frac{1}{5}}\bigr), \tag{3.6}$$

$$\int_{-\tau}^{\tau} \bigl|F_k(\alpha)\bigr|^2 \,\mathrm{d}\alpha \ll N^{\frac{2}{k}-1}. \tag{3.7}$$

Proof The inequality (3.6) follows from (2.1) and Lemma 3.1. The others are similar to Lemma 8 of Vaughan [2]. □

Lemma 3.3 *We have*

$$\int_{\mathfrak{C}} \left| \prod_{i=1}^{5} F_i(\alpha) - \prod_{i=1}^{5} I_i(\alpha) \right| K_\nu(\alpha) \,\mathrm{d}\alpha \ll \nu^2 N^{\frac{77}{60}} \exp\bigl(-L^{-\frac{1}{5}}\bigr). \tag{3.8}$$

Proof Note that

$$\begin{aligned} &\prod_{i=1}^{5} F_i(\alpha) - \prod_{i=1}^{5} I_i(\alpha) \\ &\quad = \bigl(F_1(\alpha) - I_1(\alpha)\bigr) \prod_{i=2}^{5} F_i(\alpha) + I_1(\alpha)\bigl(F_2(\alpha) - I_2(\alpha)\bigr) \prod_{i=3}^{5} F_i(\alpha) \end{aligned}$$

$$+ I_1(\alpha)I_2(\alpha)\big(F_3(\alpha) - I_3(\alpha)\big)F_4(\alpha)F_5(\alpha) + \prod_{i=1}^{3} I_i(\alpha)\big(F_4(\alpha) - I_4(\alpha)\big)F_5(\alpha)$$
$$+ \prod_{i=1}^{4} I_i(\alpha)\big(F_5(\alpha) - I_5(\alpha)\big).$$

Then by (2.7), (3.1), Lemma 3.2,

$$\begin{aligned}
&\int_{\mathfrak{C}} \left| \big(F_1(\alpha) - I_1(\alpha)\big) \prod_{i=2}^{5} F_i(\alpha) \right| K_\nu(\alpha)\, \mathrm{d}\alpha \\
&\quad \ll \nu^2 N^{\frac{47}{60}} \int_{-\tau}^{\tau} \big| \big(B_1(\alpha) - J_1(\alpha)\big) F_2(\alpha) \big| \, \mathrm{d}\alpha \\
&\quad \ll \nu^2 N^{\frac{47}{60}} \left(\int_{-\tau}^{\tau} \big| \big(B_1(\alpha) - J_1(\alpha)\big) \big|^2 \, \mathrm{d}\alpha \right)^{\frac{1}{2}} \left(\int_{-\tau}^{\tau} \big| F_2(\alpha) \big|^2 \, \mathrm{d}\alpha \right)^{\frac{1}{2}} \\
&\quad \ll \nu^2 N^{\frac{47}{60}} \left(\int_{-\tau}^{\tau} \big(\big| \big(B_1(\alpha) \big|^2 + \big| J_1(\alpha)\big) \big|^2 \big) \, \mathrm{d}\alpha \right)^{\frac{1}{2}} \\
&\quad \ll \nu^2 N^{\frac{77}{60}} \exp\big(-L^{-\frac{1}{5}}\big).
\end{aligned}$$

The other cases are similar, and the proof of Lemma 3.3 is completed. □

Lemma 3.4 *We have*

$$\int_{|\alpha|>\tau} \left| \prod_{i=1}^{5} I_i(\alpha) \right| K_\nu(\alpha)\, \mathrm{d}\alpha \ll \nu^2 N^{\frac{77}{60} - 4\theta}. \tag{3.9}$$

It follows from (2.7) and (3.3).

Lemma 3.5 *We have*

$$\int_{-\infty}^{+\infty} \prod_{j=1}^{5} I_j(\alpha) e(\alpha\eta) K_\nu(\alpha)\, \mathrm{d}\alpha \gg \nu^2 N^{\frac{77}{60}}. \tag{3.10}$$

Proof To prove (3.10), we write the left side as

$$\int_{\delta N}^{N} \int_{(\delta N)^{\frac{1}{2}}}^{N^{\frac{1}{2}}} \cdots \int_{(\delta N)^{\frac{1}{5}}}^{N^{\frac{1}{5}}} \int_{-\infty}^{+\infty} e\left(\alpha \left(\eta + \sum_{j=1}^{5} \lambda_j y_j^j \right) \right) K_\nu(\alpha)\, \mathrm{d}\alpha\, \mathrm{d}y_1\, \mathrm{d}y_2 \cdots \mathrm{d}y_5,$$

which, by (2.7), is

$$\int_{\delta N}^{N} \int_{(\delta N)^{\frac{1}{2}}}^{N^{\frac{1}{2}}} \cdots \int_{(\delta N)^{\frac{1}{5}}}^{N^{\frac{1}{5}}} \max\left(0, \nu - \left| \eta + \sum_{j=1}^{5} \lambda_j y_j^j \right| \right) \mathrm{d}y_1\, \mathrm{d}y_2 \cdots \mathrm{d}y_5. \tag{3.11}$$

We let $z_k = y_k^k$, $k = 1, 2, 3, 4, 5$, then the integral (3.11) can be written as

$$\frac{1}{120} \int_{\delta N}^{N} \cdots \int_{\delta N}^{N} z_2^{-\frac{1}{2}} z_3^{-\frac{2}{3}} z_4^{-\frac{3}{4}} z_5^{-\frac{4}{5}} \max\left(0, \nu - \left| \eta + \sum_{j=1}^{5} \lambda_j z_j \right| \right) \mathrm{d}z_1 \cdots \mathrm{d}z_5. \tag{3.12}$$

Since λ_1, λ_2, λ_3, λ_4, and λ_5 are not all of the same sign, we may assume without loss of generality that $\lambda_1 < 0$, $\lambda_2 > 0$. Consider the region

$$\mathcal{B} = \left\{(z_2, z_3, z_4, z_5) : \delta^{\frac{1}{2}} N \le z_2 \le 2\delta^{\frac{1}{2}} N, \delta N \le z_j \le 2\delta N\ (j = 3, 4, 5)\right\}.$$

Then, for δ sufficiently small and large N, whenever $(z_2, z_3, z_4, z_5) \in \mathcal{B}$ one has

$$2\delta N < -(\lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4 + \lambda_5 z_5)\lambda_1^{-1} < \frac{1}{2} N$$

and so every z_1 with $|\lambda_1 z_1 + \cdots + \lambda_5 z_5 + \eta| \le \frac{1}{2}\nu$ satisfies $\delta N < z_1 < N$. Therefore the integral (3.12) is greater than

$$\frac{1}{480}\nu^2 \int_{\mathcal{B}} z_2^{-\frac{1}{2}} z_3^{-\frac{2}{3}} z_4^{-\frac{3}{4}} z_5^{-\frac{4}{5}}\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \mathrm{d}z_5 \gg \nu^2 N^{\frac{77}{60}}.$$

This completes the proof of Lemma 3.5. □

Together with Lemmas 3.3, 3.4, 3.5, we have

$$J(\mathfrak{C}) = \int_{\mathfrak{C}} \prod_{j=1}^{5} F_j(\alpha) e(\alpha\eta) K_\nu(\alpha)\, \mathrm{d}\alpha \gg \nu^2 N^{\frac{77}{60}}. \tag{3.13}$$

4 The intermediate region

Lemma 4.1 *We have*

$$\int_{-\infty}^{+\infty} \left|F_j(\alpha)\right|^{2^j} K_\nu(\alpha)\, \mathrm{d}\alpha \ll N^{\frac{2^j}{j} - 1 + \varepsilon}, \quad j = 2, 3, 4, 5, \tag{4.1}$$

$$\int_{-\infty}^{+\infty} \left|F_1(\alpha)\right|^{2} K_\nu(\alpha)\, \mathrm{d}\alpha \ll NL. \tag{4.2}$$

Proof By (2.7), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left|F_2(\alpha)\right|^4 K_\nu(\alpha)\, \mathrm{d}\alpha \\
&\quad = \sum_{(\delta N)^{\frac{1}{2}} \le p_1, p_2, p_3, p_4 \le N^{\frac{1}{2}}} \prod_{i=1}^{4} \log p_i \max\left(0, \nu - \left|\lambda_2\left(p_1^2 + p_2^2 - p_3^2 - p_4^2\right)\right|\right) \\
&\quad \ll L^4 \sum_{(\delta N)^{\frac{1}{2}} \le p_1, p_2, p_3, p_4 \le N^{\frac{1}{2}}} \max\left(0, \nu - \left|\lambda_2\left(p_1^2 + p_2^2 - p_3^2 - p_4^2\right)\right|\right).
\end{aligned}$$

Since N is large, $|\lambda_2(p_1^2 + p_2^2 - p_3^2 - p_4^2)| < \nu$ if and only if $p_1^2 + p_2^2 = p_3^2 + p_4^2$. Thus, by Hua's inequality,

$$\int_{-\infty}^{+\infty} \left|F_2(\alpha)\right|^4 K_\nu(\alpha)\, \mathrm{d}\alpha \ll \nu N^{1+\varepsilon}.$$

The proofs of the cases $j = 3, 4, 5$ and (4.2) are similar. □

Lemma 4.2 *We have*

$$\int_{-\infty}^{+\infty} \left|F_2(\alpha)\right|^2 \left|F_4(\alpha)\right|^4 K_\nu(\alpha)\,\mathrm{d}\alpha \ll \nu N^{1+\varepsilon}. \tag{4.3}$$

Proof By (2.7), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left|F_2(\alpha)\right|^2 \left|F_4(\alpha)\right|^4 K_\nu(\alpha)\,\mathrm{d}\alpha \\
&\quad \ll L^6 \sum_{\substack{(\delta N)^{\frac{1}{2}} \le p_1,p_2 \le N^{\frac{1}{2}} \\ (\delta N)^{\frac{1}{4}} \le p_3,p_4,p_5,p_6 \le N^{\frac{1}{4}}}} \max\bigl(0, \nu - \bigl|\lambda_2\bigl(p_1^2 - p_2^2\bigr) - \lambda_4\bigl(p_3^4 + p_4^4 - p_5^4 - p_6^4\bigr)\bigr|\bigr) \\
&\quad \ll \nu L^6 R(N),
\end{aligned}$$

where $R(N)$ is the number of the solutions of the equation

$$\begin{aligned}
&\lambda_2\bigl(p_1^2 - p_2^2\bigr) = \lambda_4\bigl(p_3^4 + p_4^4 - p_5^4 - p_6^4\bigr), \\
&(\delta N)^{\frac{1}{2}} \le p_1, p_2 \le N^{\frac{1}{2}}, \qquad (\delta N)^{\frac{1}{4}} \le p_3, p_4, p_5, p_6 \le N^{\frac{1}{4}}.
\end{aligned}$$

Then we have

$$R(N) \ll N^{\frac{1}{2}} \sum_{\substack{(\delta N)^{\frac{1}{4}} \le p_3,p_4,p_5,p_6 \le N^{\frac{1}{4}} \\ p_3^4+p_4^4-p_5^4-p_6^4=0}} 1 + \sum_{\substack{(\delta N)^{\frac{1}{4}} \le p_3,p_4,p_5,p_6 \le N^{\frac{1}{4}} \\ p_3^4+p_4^4-p_5^4-p_6^4\neq 0}} d\bigl(\bigl|p_3^4 + p_4^4 - p_5^4 - p_6^4\bigr|\bigr),$$

where $d(n)$ is the divisor function. Now (4.3) follows from [3], (2.1). □

Lemma 4.3 ([4]) *Suppose that* $(a,q) = 1$, $|\alpha - a/q| \le q^{-2}$, *then*

$$\sum_{1 \le p \le X} (\log p) e(p\alpha) \ll (\log X)^5 \bigl(X^{1/2} q^{1/2} + X^{4/5} + X q^{-1/2}\bigr).$$

Lemma 4.4 ([5]) *Suppose that* $(a,q) = 1$, $|\alpha - a/q| \le q^{-2}$, $\phi(x) = \alpha x^k + \alpha_1 x^{k-1} + \cdots + \alpha_{k-1} x + \alpha_k$ $(k \ge 2)$, *then*

$$\sum_{1 \le p \le X} (\log p) e\bigl(\phi(p)\bigr) \ll X^{1+\varepsilon} \bigl(q^{-1} + X^{-1/2} + q X^{-k}\bigr)^{4^{1-k}}.$$

Lemma 4.5 *For* $\tau < |\alpha| \le P$, *we have*

$$V(\alpha) := \min\bigl(F_1(\alpha), F_2(\alpha)^2\bigr) \ll N^{1-\frac{\theta}{2}+\varepsilon}.$$

Proof Let $\tau < |\alpha| \le P$, we choose a_j, q_j $(j = 1,2)$ so that $|\lambda_j \alpha - a_j/q_j| \le Q^{-1} q_j^{-1}$ with $(a_j, q_j) = 1$ and $1 \le q_j \le Q$. By the method of Davenport and Heilbronn (see Lemma 11 of [6]), we have $\max(q_1, q_2) \ge P$. Then Lemma 4.5 follows from Lemmas 4.3 and 4.4. □

Lemma 4.6 *We have*

$$J(\mathfrak{D}) = \int_{\mathfrak{D}} \prod_{j=1}^{5} F_j(\alpha) e(\alpha\eta) K_\nu(\alpha)\,\mathrm{d}\alpha \ll \nu^2 N^{\frac{77}{60}-(\frac{\theta}{32}-\sigma)+\varepsilon}. \tag{4.4}$$

Proof By Lemmas 4.1, 4.2, 4.5, and Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathfrak{D}} \left| \prod_{j=1}^{5} F_j(\alpha) e(\alpha\eta) K_\nu(\alpha) \right| \mathrm{d}\alpha \\
&\quad \ll V(\alpha)^{\frac{1}{16}} \int_{-\infty}^{+\infty} \left| \left(F_1(\alpha)^{\frac{15}{16}} F_2(\alpha) + F_1(\alpha) F_2(\alpha)^{\frac{7}{8}} \right) \prod_{j=3}^{5} F_j(\alpha) \right| K_\nu(\alpha)\,\mathrm{d}\alpha \\
&\quad \ll V(\alpha)^{\frac{1}{16}} \left(\int_{-\infty}^{+\infty} \left|F_1(\alpha)\right|^2 K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{15}{32}} \left(\int_{-\infty}^{+\infty} \left|F_2(\alpha)\right|^4 K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{1}{8}} \\
&\qquad \times \left(\int_{-\infty}^{+\infty} \left|F_2(\alpha)^2 F_4(\alpha)^4\right| K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{1}{4}} \left(\int_{-\infty}^{+\infty} \left|F_3(\alpha)\right|^8 K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{1}{8}} \\
&\qquad \times \left(\int_{-\infty}^{+\infty} \left|F_5(\alpha)\right|^{32} K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{1}{32}} + V(\alpha)^{\frac{1}{16}} \left(\int_{-\infty}^{+\infty} \left|F_1(\alpha)\right|^2 K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_{-\infty}^{+\infty} \left|F_2(\alpha)\right|^4 K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{3}{32}} \left(\int_{-\infty}^{+\infty} \left|F_2(\alpha)^2 F_4(\alpha)^4\right| K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{1}{4}} \\
&\qquad \times \left(\int_{-\infty}^{+\infty} \left|F_3(\alpha)\right|^8 K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{1}{8}} \left(\int_{-\infty}^{+\infty} \left|F_5(\alpha)\right|^{32} K_\nu(\alpha)\,\mathrm{d}\alpha \right)^{\frac{1}{32}} \\
&\quad \ll \nu N^{\frac{77}{60}-\frac{\theta}{32}+\varepsilon} \ll \nu^2 N^{\frac{77}{60}-(\frac{\theta}{32}-\sigma)+\varepsilon}.
\end{aligned}$$

□

5 The trivial region

Lemma 5.1 *Let $G(\alpha) = \sum e(\alpha f(x_1,\ldots,x_m))$, where f is any real function and the summation is over any finite set of values of $x_1,\ldots,x_m$. Then, for any $A > 4$, we have*

$$\int_{|\alpha|>A} \left|G(\alpha)\right|^2 K_\nu(\alpha)\,\mathrm{d}\alpha \le \frac{16}{A} \int_{-\infty}^{+\infty} \left|G(\alpha)\right|^2 K_\nu(\alpha)\,\mathrm{d}\alpha.$$

This is Lemma 2 *of* [7].

Lemma 5.2 *We have*

$$J(\mathfrak{c}) = \int_{\mathfrak{c}} \prod_{j=1}^{5} F_j(\alpha) e(\alpha\eta) K_\nu(\alpha)\,\mathrm{d}\alpha \ll \nu^2 N^{\frac{77}{60}-(\theta-\sigma)+\varepsilon}.$$

Proof By Lemmas 4.1, 4.2, 5.1, and Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathfrak{c}} \left| \prod_{j=1}^{5} F_j(\alpha) e(\alpha\eta) K_\nu(\alpha) \right| \mathrm{d}\alpha \\
&\quad \ll \frac{1}{P} \int_{-\infty}^{+\infty} \prod_{j=1}^{5} \left|F_j(\alpha)\right| K_\nu(\alpha)\,\mathrm{d}\alpha
\end{aligned}$$

$$
\begin{aligned}
&\ll \frac{1}{P}\max\left(\left|F_5(\alpha)\right|\right)\left(\int_{-\infty}^{+\infty}\left|F_1(\alpha)\right|^2 K_\nu(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\left|F_2(\alpha)\right|^4 K_\nu(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{8}}\\
&\quad\times\left(\int_{-\infty}^{+\infty}\left|F_2(\alpha)^2F_4(\alpha)^4\right|K_\nu(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{-\infty}^{+\infty}\left|F_3(\alpha)\right|^8 K_\nu(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{8}}\\
&\ll \nu N^{\frac{77}{60}-\theta+\varepsilon} \ll \nu^2 N^{\frac{77}{60}-(\theta-\sigma)+\varepsilon}. \qquad \square
\end{aligned}
$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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